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Elastic Field of a Dilatational Cylindrical Inclusion in an Elastically Isotropic Half-Space

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Abstract. In this article, a new solution to the elasticity boundary-value problem for a dilatational cylindrical inclusion embedded in an elastically isotropic half-space is presented. To solve this problem, the results for the infinitesimally thin dilatational disk in an elastically isotropic half-space, are explored. For displacements, strains, and stresses of a dilatational cylindrical inclusion, the analytical expressions are obtained with Lipschitz-Hankel integrals. The comparison of the found solution with previously known one, is given.

1. INTRODUCTION

Inclusions along with dislocations in an elastic medium have been studied since the 1950s by Eshelby [1], Mura [2], Teodosiu [3], and other authors. Based on those basic results, there have been many studies on elastic fields of inclusions in the elastic media with special attention to the phase boundaries and interfaces [4,5].

The stressors, e.g., inclusions, which are located in an elastic medium, act as the source of the elastic field inside the solid, and thus affect the physical and mechanical properties of the material. First, the stored elastic energy associated with stressors can be released giving rise to the formation of various defects [6–9], such as dislocations, cracks, etc. Second, stressors induce elastic distortions changing interatomic distances from their equilibrium values and thus changing material properties [10,11]. The influence of the elastic field inside the material that affects the physical and mechanical properties of the materials have been shown in Ref. [12]. Theoretical and experimental studies on the influence of the elastic field of the stressor on the mechanical properties and electronic structure of semiconductors have been studied in Refs. [12,13].

In this study, we focus on the stressor with circular symmetry — a finite dilatational cylindrical inclusion (DCI), which is placed near the free surface in an isotropic elastic half-space and with the symmetry axis normal to the free surface.

In Section 2 we provide background knowledge about the elastic properties of the infinitesimally thin dilatational disk (ITDD) in an elastic half-space [10]. In Section 3 we formulate the statement of the elastic problem of a cylindrical inclusion placed near a free surface in an elastically isotropic half-space. In Section 4 we find the expression for the elastic field of the cylindrical inclusion. In Sections 5 and 6 we will discuss and conclude the results found.

2. BACKGROUND

2.1. Dilatational cylindrical inclusion

Consider DCI subjected to a dilatational eigenstrain which has the following form:

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$\varepsilon^*_{ii} = \varepsilon^*\delta(\Omega)$, with $i = x, y, z$ or r, φ, z, (1)

where ε^* is the magnitude of the eigenstrain components, $\delta(\Omega) = \begin{cases} 1, & R \in \Omega \\ 0, & R \notin \Omega \end{cases}$ is Dirac delta-function, Ω is the three-dimensional (3D) region occupied by the DCI, and r, φ, z are cylindrical coordinates (see Fig. 1)

By definition, any inclusion is a 3D defect. Therefore, it can be assembled from defects of lower dimension — infinitesimally thin dilatational disks (ITDDs). The eigenstrain of the dilatational disk can be written as [11]

$$^{\mathrm{ITDD}}\varepsilon^*_{ii} = bH\left(1 - \frac{r}{c}\right)\delta(z - z_0), \tag{2}$$

where b is a dimensional factor characterizing the local magnitude of dilatation, $H(\zeta) = \begin{cases} 1, & \zeta \geq 0 \\ 0, & \zeta < 0 \end{cases}$ is Heaviside step-function, $\delta(z)$ is Dirac delta-function and no Einstein summation rule is used.

Distributing the disks along their symmetry axis with a constant density ρ, we form the DCI with eigenstrain (1):

$$^{\mathrm{DCI}}\varepsilon^*_{ii} = \int_{z_1}^{z_2} {}^{\mathrm{ITDD}}\varepsilon^*_{ii}(r, \varphi, z, z_0)\rho dz_0$$

$$= \int_{z_1}^{z_2} b\rho H\left(1 - \frac{r}{c}\right)\delta(z - z_0)dz_0 = b\rho\delta(\Omega) = \varepsilon^*\delta(\Omega), \tag{3}$$

where it is assumed that $\varepsilon^* = b\rho$. Here, we accept that distribution density ρ of the disks along the z-axis is constant. In general, ρ may depend on z_0 (position of ITDD).

It follows from the above that by integrating the field of total displacements of the disks $^{\mathrm{ITDD}}u_i^{\mathrm{hs}}$ (in the elastic half-space), we obtain the total displacements of DCI $^{\mathrm{DCI}}u_i^{\mathrm{hs}}$:

$$^{\mathrm{DCI}}u_i^{\mathrm{hs}} = \int_{z_1}^{z_2} {}^{\mathrm{ITDD}}u_i^{\mathrm{hs}}(r, \varphi, z, z_0)\rho dz_0. \tag{4}$$

Eq. (4) allows to calculate the displacements for finite cylindrical, spherical inclusions, and truncated cone in the elastic space.

2.2. Infinitesimally thin dilatational disk in the elastic half-space

Displacements of ITDD in the elastic half-space were found in Ref. [10] in a cylindrical coordinate system (r, φ, z), $0 \leq r \leq \infty$, $0 \leq \varphi \leq 2\pi$, $-\infty \leq z \leq +\infty$.

$$^{\mathrm{ITDD}}u_r^{\mathrm{hs}} = \frac{(1+\nu)b}{2(1-\nu)}\Big[J^{(1)}(1,1;0) - J^{(2)}(1,1;0)$$
$$+\mathrm{sgn}(z)\left(\frac{2|z|}{c}J^{(3)}(1,1;1) - 4(1-\nu)J^{(3)}(1,1;0)\right)\Big], \tag{5a}$$

$$^{\mathrm{ITDD}}u_\varphi^{\mathrm{hs}} = 0, \tag{5b}$$

$$^{\mathrm{ITDD}}u_z^{\mathrm{hs}} = \frac{(1+\nu)b}{2(1-\nu)}\Big[\mathrm{sgn}(z - z_0)J^{(1)}(1,0;0) - \mathrm{sgn}(z + z_0)$$
$$\times J^{(2)}(1,0;0) + \frac{2|z|}{c}J^{(3)}(1,0;1) - 2(2\nu - 1)J^{(3)}(1,0;0)\Big]. \tag{5c}$$

Here: $J^{(l)}(m, n; p)$ are the Lipschitz-Hankel integrals:

$$J^{(l)}(m, n; p) = \int_0^\infty J_m(\kappa)J_n(\kappa\frac{r}{c})\kappa^p e^{-\kappa\xi_l}d\kappa,$$

where J_m and J_n are Bessel functions of the first kind, $l = 1, 2, 3$, c is radius of ITDD, and:

$$\xi_1 = \frac{|z - z_0|}{c},\ \xi_2 = \frac{|z + z_0|}{c},\ \xi_3 = \frac{|z| + |z_0|}{c}.$$

3. STATEMENT OF THE PROBLEM AND THE METHOD OF SOLUTION

As shown in Fig. 1, a DCI of radius c and height h is placed at a distance d (to the center of the cylinder) from the free surface in the elastically isotropic half-space. As it was mentioned in Section 2.1, the DCI can be thought to be composed of coaxial ITDDs distributed with the density ρ along the z-axis; see Fig. 1. The total displacements of the DCI in the half-space are equal to the sum of the displacements of the ITDDs in the elastic half-space. We distribute ITDDs along the z-axis uniformly from z_1 to z_2 and find total displacements of the DCI, as follows:

$$^{\mathrm{DCI}}u_r^{\mathrm{hs}} = \int_{z_1}^{z_2} {}^{\mathrm{ITDD}}u_r^{\mathrm{hs}}(r, \varphi, z, z_0)\rho dz_0$$
$$= \frac{(1+\nu)b\rho}{2(1-\nu)}\int_{z_1}^{z_2}\Big[J^{(1)}(1,1;0) - J^{(2)}(1,1;0)$$
$$+\mathrm{sgn}(z)\left(\frac{2|z|}{c}J^{(3)}(1,1;1) - 4(1-\nu)J^{(3)}(1,1;0)\right)\Big]dz_0, \tag{6a}$$

$$^{\mathrm{DCI}}u_\varphi^{\mathrm{hs}} = 0, \tag{6b}$$

$$^{\mathrm{DCI}}u_z^{\mathrm{hs}} = \int_{z_1}^{z_2} {}^{\mathrm{ITDD}}u_z^{\mathrm{hs}}(r, \varphi, z, z_0)\rho dz_0$$
$$= \frac{(1+\nu)b\rho}{2(1-\nu)}\int_{z_1}^{z_2}\Big[\mathrm{sgn}(z - z_0)J^{(1)}(1,0;0) - \mathrm{sgn}(z + z_0)J^{(2)}(1,0;0)$$

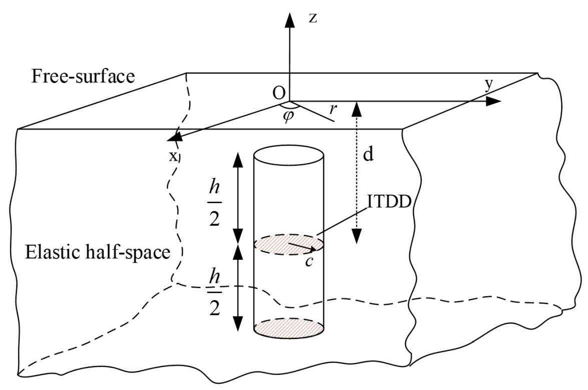

Fig. 1. Cylinder in elastically isotropic half-space.

$$+\frac{2|z|}{c}J^{(3)}(1,0;1)-2(2\nu-1)J^{(3)}(1,0;0)\Big]dz_0, \tag{6c}$$

where $z_1=-\left(d+\frac{h}{2}\right)$, $z_2=-\left(d-\frac{h}{2}\right)$.

The DCI is perfectly aligned with the surrounding matrix. The free surface is flat, smooth, and stress-free. The elasticity of the material is linear and, thus, obeys Hooke's law [12].

$$\sigma_{ij}=2G\left(\varepsilon_{ij}+\frac{\nu}{1-2\nu}\Delta\,\delta_{ij}\right), i, j, k=\begin{cases}x, y, z,\\ \text{or}\\ r, \varphi, z,\end{cases}, \tag{7}$$

where G is shear modulus and ν is Poisson's ratio, δ_{ij} is Kronecker delta $\delta_{ij}=\begin{cases}1, i=j\\ 0, i\neq j\end{cases}$, and Einstein summation rule is applied. By solving the Eqs. (6), we obtain the displacements ${}^{\text{DCI}}u_i^{\text{hs}}$ of the DCI in an elastic isotropic half-space. From that result, we can determine the strains ${}^{\text{DCI}}\varepsilon_{ij}^{\text{hs}}$ and stresses ${}^{\text{DCI}}\sigma_{ij}^{\text{hs}}$ of DCI in half-space.

4. ELASTICITY OF DILATATIONAL CYLINDRICAL INCLUSIONS IN AN ELASTICALLY ISOTROPIC HALF-SPACE

4.1. Displacements of cylindrical inclusion in the half-space

First, we calculate the integrals $\int_{z_1}^{z_2}J^{(l)}(m,n;p)\,dz_0$ (with l = 1, 2, 3), see Appendix A, and then substitute them into Eqs. (6) to get the total displacements inside and outside the DCI (denoted by indices "in" and "out", respectively), in terms of the Lipschitz-Hankel integrals.

4.1.1. Inside the cylinder ($z_1<z<z_2$ and $r\le c$)

$$\begin{aligned}{}^{\text{DCI, in}}u_r^{\text{hs}}=\frac{(1+\nu)c\varepsilon^*}{2(1-\nu)}\Big[&\frac{r}{c}-J^{(4)}(1,1;-1)-J^{(5)}(1,1;-1)\\&+(4\nu-3)\big(J^{(6)}(1,1;-1)-J^{(7)}(1,1;-1)\big)\\&-\frac{2z}{c}\big(J^{(6)}(1,1;0)-J^{(7)}(1,1;0)\big)\Big],\end{aligned} \tag{8a}$$

$${}^{\text{DCI, in}}u_\varphi^{\text{hs}}=0, \tag{8b}$$

$$\begin{aligned}{}^{\text{DCI, in}}u_z^{\text{hs}}=\frac{(1+\nu)c\varepsilon^*}{2(1-\nu)}\Big[&-J^{(4)}(1,0;-1)+J^{(5)}(1,0;-1)\\&+\frac{2z}{c}\big(J^{(6)}(1,0;0)-J^{(7)}(1,0;0)\big)\\&+(4\nu-3)\big(J^{(6)}(1,0;-1)-J^{(7)}(1,0;-1)\big)\Big].\end{aligned} \tag{8c}$$

4.1.2. Outside the cylinder ($z<z_1$, or $z>z_2$, or $z_1<z<z_2$ and $r>c$)

For $z<z_1$:

$$\begin{aligned}{}^{\text{DCI, out}}u_r^{\text{hs}}=\frac{(1+\nu)c\varepsilon^*}{2(1-\nu)}\Big[&J^{(4)}(1,1;-1)-J^{(5)}(1,1;-1)\\&+(4\nu-3)\big(J^{(6)}(1,1;-1)-J^{(7)}(1,1;-1)\big)\\&-\frac{2z}{c}\big(J^{(6)}(1,1;0)-J^{(7)}(1,1;0)\big)\Big],\end{aligned} \tag{9a}$$

$$\begin{aligned}{}^{\text{DCI, out}}u_z^{\text{hs}}=\frac{(1+\nu)c\varepsilon^*}{2(1-\nu)}\Big[&-J^{(4)}(1,0;-1)+J^{(5)}(1,0;-1)\\&+\frac{2z}{c}\big(J^{(6)}(1,0;0)-J^{(7)}(1,0;0)\big)\\&+(4\nu-3)\big(J^{(6)}(1,0;-1)-J^{(7)}(1,0;-1)\big)\Big].\end{aligned} \tag{9b}$$

For $z>z_2$:

$$\begin{aligned}{}^{\text{DCI, out}}u_r^{\text{hs}}=\frac{(1+\nu)c\varepsilon^*}{2(1-\nu)}\Big[&-J^{(4)}(1,1;-1)+J^{(5)}(1,1;-1)\\&+(4\nu-3)\big(J^{(6)}(1,1;-1)-J^{(7)}(1,1;-1)\big)\\&-\frac{2z}{c}\big(J^{(6)}(1,1;0)-J^{(7)}(1,1;0)\big)\Big],\end{aligned} \tag{9c}$$

$$\begin{aligned}{}^{\text{DCI, out}}u_z^{\text{hs}}=\frac{(1+\nu)c\varepsilon^*}{2(1-\nu)}\Big[&-J^{(4)}(1,0;-1)+J^{(5)}(1,0;-1)\\&+\frac{2z}{c}\big(J^{(6)}(1,0;0)-J^{(7)}(1,0;0)\big)\\&+(4\nu-3)\big(J^{(6)}(1,0;-1)-J^{(7)}(1,0;-1)\big)\Big].\end{aligned} \tag{9d}$$

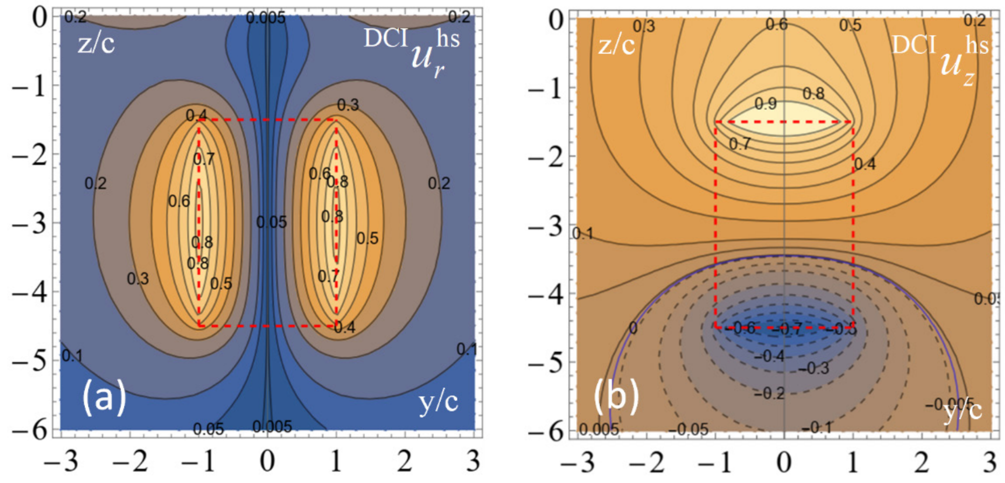

Fig. 2. Displacement maps of a cylindrical inclusion in an elastic half-space. Displacements are given in units $(1+\nu)\varepsilon^*/[2(1-\nu)]$, and the coordinates are normalized to the cylinder radius c. Parameters used for plots: the cylinder center coordinate is $z_c = -d = -3c$, the height of the cylinder is $h = 3c$, and Poisson ratio is $\nu = 0.234$.

For $z_1 < z < z_2$ and $r > c$:

$$ {}^{\rm DCI,\,out}u_r^{\rm hs} = \frac{(1+\nu)c\varepsilon^*}{2(1-\nu)}\Big[\frac{c}{r} - J^{(4)}(1,1;-1) - J^{(5)}(1,1;-1) + (4\nu-3)\big(J^{(6)}(1,1;-1) - J^{(7)}(1,1;-1)\big) - \frac{2z}{c}\big(J^{(6)}(1,1;0) - J^{(7)}(1,1;0)\big)\Big]. \tag{9e} $$

$$ {}^{\rm DCI,\,out}u_z^{\rm hs} = \frac{(1+\nu)c\varepsilon^*}{2(1-\nu)}\Big[-J^{(4)}(1,0;-1) + J^{(5)}(1,0;-1) + \frac{2z}{c}\big(J^{(6)}(1,0;0) - J^{(7)}(1,0;0)\big) + (4\nu-3)\big(J^{(6)}(1,0;-1) - J^{(7)}(1,0;-1)\big)\Big]. \tag{9f} $$

For $(z < z_1,\ \text{or}\ z > z_2,\ \text{or}\ z_1 < z < z_2\ \text{and}\ r > c)$:

$$ {}^{\rm DCI,\,out}u_\varphi^{\rm hs} = 0. \tag{9g} $$

Here $J^{(l)}(m,n;p) = \int_0^\infty J_m(\kappa)J_n(\kappa\frac{r}{c})\kappa^p e^{-\kappa\xi_l}d\kappa$, with $l = 4, 5, 6, 7$, and:

$$ \xi_4 = \frac{|z-z_1|}{c},\quad \xi_5 = \frac{|z-z_2|}{c},\quad \xi_6 = \frac{|z+z_1|}{c},\quad \xi_7 = \frac{|z+z_2|}{c}, $$

$$ z_1 = -\left(d + \frac{h}{2}\right),\quad z_2 = -\left(d - \frac{h}{2}\right). $$

The contour plots for the u_r and u_z components of the displacements are shown in Fig. 2. The Fig. 2 shows the influence of the free surface on displacements of DCI in half-space compared to the infinite space case [13].

4.2. Strains of cylindrical inclusion in the half-space

Elastic strains are determined from total displacements (8,9) as follows [2]:

$$ {}^{\rm DCI}\varepsilon_{rr}^{\rm hs} = \begin{cases} \dfrac{\partial\,{}^{\rm DCI,\,in}u_r^{\rm hs}}{\partial r} - \varepsilon^*, & \text{inside DCI,} \\ \dfrac{\partial\,{}^{\rm DCI,\,out}u_r^{\rm hs}}{\partial r}, & \text{outside DCI,} \end{cases} \tag{10a} $$

$$ {}^{\rm DCI}\varepsilon_{\varphi\varphi}^{\rm hs} = \begin{cases} \dfrac{\partial\,{}^{\rm DCI,\,in}u_r^{\rm hs}}{\partial r} - \varepsilon^*, & \text{inside DCI,} \\ \dfrac{\partial\,{}^{\rm DCI,\,out}u_r^{\rm hs}}{\partial r}, & \text{outside DCI,} \end{cases} \tag{10b} $$

$$ {}^{\rm DCI}\varepsilon_{zz}^{\rm hs} = \begin{cases} \dfrac{\partial\,{}^{\rm DCI,\,in}u_z^{\rm hs}}{\partial z} - \varepsilon^*, & \text{inside DCI,} \\ \dfrac{\partial\,{}^{\rm DCI,\,out}u_z^{\rm hs}}{\partial z}, & \text{outside DCI,} \end{cases} \tag{10c} $$

$$ {}^{\rm DCI}\varepsilon_{rz}^{\rm hs} = \begin{cases} \dfrac{1}{2}\left(\dfrac{\partial\,{}^{\rm DCI,\,in}u_r^{\rm hs}}{\partial z} + \dfrac{\partial\,{}^{\rm DCI,\,in}u_z^{\rm hs}}{\partial r}\right), & \text{inside DCI,} \\ \dfrac{1}{2}\left(\dfrac{\partial\,{}^{\rm DCI,\,out}u_r^{\rm hs}}{\partial z} + \dfrac{\partial\,{}^{\rm DCI,\,out}u_z^{\rm hs}}{\partial r}\right), & \text{outside DCI,} \end{cases} \tag{10d} $$

Detailed expressions (10) of the strains are listed in Appendix B.

The elastic dilatation (trace of strain tensor) of the DCI is found with the help of Eqs. (8–10):

$$ {}^{\text{DCI, in}}\Delta^{\text{hs}} = {}^{\text{DCI, in}}\varepsilon_{rr}^{\text{hs}} + {}^{\text{DCI, in}}\varepsilon_{\varphi\varphi}^{\text{hs}} + {}^{\text{DCI, in}}\varepsilon_{zz}^{\text{hs}} $$

$$ = \frac{2(2\nu-1)\varepsilon^*}{1-\nu} + \frac{2(1+\nu)\varepsilon^*}{1-\nu}(2\nu-1)\left(J^{(6)}(1,0;0) - J^{(7)}(1,0;0)\right), \quad (11a) $$

$$ {}^{\text{DCI, out}}\Delta^{\text{hs}} = {}^{\text{DCI, out}}\varepsilon_{rr}^{\text{hs}} + {}^{\text{DCI, out}}\varepsilon_{\varphi\varphi}^{\text{hs}} + {}^{\text{DCI, out}}\varepsilon_{zz}^{\text{hs}} $$

$$ = \frac{2(1+\nu)\varepsilon^*}{1-\nu}(2\nu-1)\left[J^{(6)}(1,0;0) - J^{(7)}(1,0;0)\right]. \quad (11b) $$

From Eqs. (11) we see that both inside and outside the cylinder, the dilatation is non-zero, and the first term in Eq. (11a) is the elastic dilatation of the DCI in infinite space; see Ref. [13]. The second term is related to the boundary condition induced by the half-space surface [2].

4.3. Stresses of dilatational cylindrical inclusion in the half-space

Stresses inside and outside of DCI are determined from Hooke's law (7) with the help of Eqs. (10) and (11) as follows [12]:

4.3.1. Inside the cylinder ($z_1 < z < z_2$ and $r \le c$)

$$ {}^{\text{DCI, in}}\sigma_{rr}^{\text{hs}} = G\frac{(1+\nu)\varepsilon^*}{1-\nu}\Bigg[\frac{c}{r}J^{(4)}(1,1;-1) + \frac{1}{r}J^{(5)}(1,1;-1) $$
$$ -J^{(4)}(1,0;0) - J^{(5)}(1,0;0) - 3\left(J^{(6)}(1,0;0) - J^{(7)}(1,0;0)\right) $$
$$ -\frac{c}{r}(4\nu-3)\left(J^{(6)}(1,1;-1) - J^{(7)}(1,1;-1)\right) $$
$$ +\frac{2z}{r}\left(J^{(6)}(1,1;0) - J^{(7)}(1,1;0)\right) $$
$$ -\frac{2z}{c}\left(J^{(6)}(1,0;1) - J^{(7)}(1,0;1)\right) - 1\Bigg], \quad (12a) $$

$$ {}^{\text{DCI, in}}\sigma_{\varphi\varphi}^{\text{hs}} = G\frac{(1+\nu)\varepsilon^*}{1-\nu}\Bigg[-\frac{c}{r}\left(J^{(4)}(1,1;-1) + J^{(5)}(1,1;-1)\right) $$
$$ -\frac{2z}{r}\left(J^{(6)}(1,1;0) - J^{(7)}(1,1;0)\right) $$
$$ +\frac{c}{r}(4\nu-3)\left(J^{(6)}(1,1;-1) - J^{(7)}(1,1;-1)\right) $$
$$ -4\nu\left(J^{(6)}(1,0;0) - J^{(7)}(1,0;0)\right) - 1\Bigg], \quad (12b) $$

$$ {}^{\text{DCI, in}}\sigma_{zz}^{\text{hs}} = G\frac{(1+\nu)\varepsilon^*}{1-\nu}\Bigg[J^{(4)}(1,0;0) + J^{(5)}(1,0;0) $$
$$ +\frac{2z}{c}\left(J^{(6)}(1,0;1) - J^{(7)}(1,0;1)\right) $$
$$ -J^{(6)}(1,0;0) + J^{(7)}(1,0;0) - 2\Bigg], \quad (12c) $$

$$ {}^{\text{DCI, in}}\sigma_{rz}^{\text{hs}} = G\frac{(1+\nu)\varepsilon^*}{1-\nu}\Bigg[J^{(4)}(1,1;0) - J^{(5)}(1,1;0) $$
$$ -J^{(6)}(1,1;0) + J^{(7)}(1,1;0) - \frac{2z}{c}\left(J^{(6)}(1,1;1) - J^{(7)}(1,1;1)\right)\Bigg]. \quad (12d) $$

4.3.2. Outside the cylinder ($z < z_1$, or $z > z_2$, or $z_1 < z < z_2$ and $r > c$)

For $z > z_2$:

$$ {}^{\text{DCI, out}}\sigma_{rr}^{\text{hs}} = G\frac{(1+\nu)\varepsilon^*}{1-\nu}\Bigg[-\frac{c}{r}\left(J^{(4)}(1,1;-1) - J^{(5)}(1,1;-1)\right) $$
$$ +J^{(4)}(1,0;0) - J^{(5)}(1,0;0) - 3\left(J^{(6)}(1,0;0) - J^{(7)}(1,0;0)\right) $$
$$ -(4\nu-3)\frac{c}{r}\left(J^{(6)}(1,1;-1) - J^{(7)}(1,1;-1)\right) $$
$$ +\frac{2z}{r}\left(J^{(6)}(1,1;0) - J^{(7)}(1,1;0)\right) $$
$$ -\frac{2z}{c}\left(J^{(6)}(1,0;1) - J^{(7)}(1,0;1)\right)\Bigg], \quad (13a) $$

$$ {}^{\text{DCI, out}}\sigma_{\varphi\varphi}^{\text{hs}} = G\frac{(1+\nu)\varepsilon^*}{1-\nu}\Bigg[\frac{c}{r}\left(J^{(4)}(1,1;-1) - J^{(5)}(1,1;-1)\right) $$
$$ +\frac{c}{r}(4\nu-3)\left(J^{(6)}(1,1;-1) - J^{(7)}(1,1;-1)\right) $$
$$ -\frac{2z}{r}\left(J^{(6)}(1,1;0) - J^{(7)}(1,1;0)\right) $$
$$ -4\nu\left(J^{(6)}(1,0;0) - J^{(7)}(1,0;0)\right)\Bigg], \quad (13b) $$

$$ {}^{\text{DCI, out}}\sigma_{zz}^{\text{hs}} = G\frac{(1+\nu)\varepsilon^*}{1-\nu}\Bigg[-J^{(4)}(1,0;0) + J^{(5)}(1,0;0) $$
$$ +\frac{2z}{c}\left(J^{(6)}(1,0;1) - J^{(7)}(1,0;1)\right) $$
$$ -\left(J^{(6)}(1,0;0) - J^{(7)}(1,0;0)\right)\Bigg]. \quad (13c) $$

For $z > z_2$:

$$ {}^{\text{DCI, out}}\sigma_{rr}^{\text{hs}} = G\frac{(1+\nu)\varepsilon^*}{1-\nu}\Bigg[\frac{c}{r}\left(J^{(4)}(1,1;-1) - J^{(5)}(1,1;-1)\right) $$
$$ -J^{(4)}(1,0;0) + J^{(5)}(1,0;0) - 3\left(J^{(6)}(1,0;0) - J^{(7)}(1,0;0)\right) $$
$$ -(4\nu-3)\frac{c}{r}\left(J^{(6)}(1,1;-1) - J^{(7)}(1,1;-1)\right) $$
$$ +\frac{2z}{r}\left(J^{(6)}(1,1;0) - J^{(7)}(1,1;0)\right) $$
$$ -\frac{2z}{c}\left(J^{(6)}(1,0;1) - J^{(7)}(1,0;1)\right)\Bigg], \quad (13d) $$

$$
{}^{\text{DCI, out}}\sigma_{\varphi\varphi}^{\text{hs}} = G\frac{(1+\nu)\varepsilon^*}{1-\nu}\Big[-\frac{c}{r}\big(J^{(4)}(1,1;-1)-J^{(5)}(1,1;-1)\big)
+\frac{c}{r}(4\nu-3)\big(J^{(6)}(1,1;-1)-J^{(7)}(1,1;-1)\big)
-\frac{2z}{r}\big(J^{(6)}(1,1;0)-J^{(7)}(1,1;0)\big)
-4\nu\big(J^{(6)}(1,0;0)-J^{(7)}(1,0;0)\big)\Big], \qquad (13e)
$$

$$
{}^{\text{DCI, out}}\sigma_{zz}^{\text{hs}} = G\frac{(1+\nu)\varepsilon^*}{1-\nu}\Big[J^{(4)}(1,0;0)-J^{(5)}(1,0;0)
+\frac{2z}{c}\big(J^{(6)}(1,0;1)-J^{(7)}(1,0;1)\big)
-\big(J^{(6)}(1,0;0)-J^{(7)}(1,0;0)\big)\Big]. \qquad (13f)
$$

For $z_1 < z < z_2$:

$$
{}^{\text{DCI, out}}\sigma_{rr}^{\text{hs}} = G\frac{(1+\nu)\varepsilon^*}{1-\nu}\Big[-\frac{c^2}{r^2}-J^{(4)}(1,0;0)-J^{(5)}(1,0;0)
+\frac{c}{r}\big(J^{(4)}(1,1;-1)+J^{(5)}(1,1;-1)\big)
-(4\nu-3)\frac{c}{r}\big(J^{(6)}(1,1;-1)-J^{(7)}(1,1;-1)\big)
-3\big(J^{(6)}(1,0;0)-J^{(7)}(1,0;0)\big)
+\frac{2z}{r}\big(J^{(6)}(1,1;0)-J^{(7)}(1,1;0)\big)
-\frac{2z}{c}\big(J^{(6)}(1,0;1)-J^{(7)}(1,0;1)\big)\Big], \qquad (13g)
$$

$$
{}^{\text{DCI, out}}\sigma_{\varphi\varphi}^{\text{hs}} = G\frac{(1+\nu)\varepsilon^*}{1-\nu}\Big[\frac{c^2}{r^2}
-\frac{c}{r}\Big(J^{(4)}(1,1;-1)+\frac{c}{r}J^{(5)}(1,1;-1)\Big)
+\frac{c}{r}(4\nu-3)\big(J^{(6)}(1,1;-1)-J^{(7)}(1,1;-1)\big)
-\frac{2z}{r}\big(J^{(6)}(1,1;0)-J^{(7)}(1,1;0)\big)
-4\nu\big(J^{(6)}(1,0;0)-J^{(7)}(1,0;0)\big)\Big], \qquad (13h)
$$

$$
{}^{\text{DCI, out}}\sigma_{zz}^{\text{hs}} = G\frac{(1+\nu)\varepsilon^*}{1-\nu}\Big[J^{(4)}(1,0;0)+J^{(5)}(1,0;0)
+\frac{2z}{c}\big(J^{(6)}(1,0;1)-J^{(7)}(1,0;1)\big)
-\big(J^{(6)}(1,0;0)-J^{(7)}(1,0;0)\big)\Big]. \qquad (13i)
$$

For $z < z_1$, or $z > z_2$, or $z_1 < z < z_2$ and $r > c$:

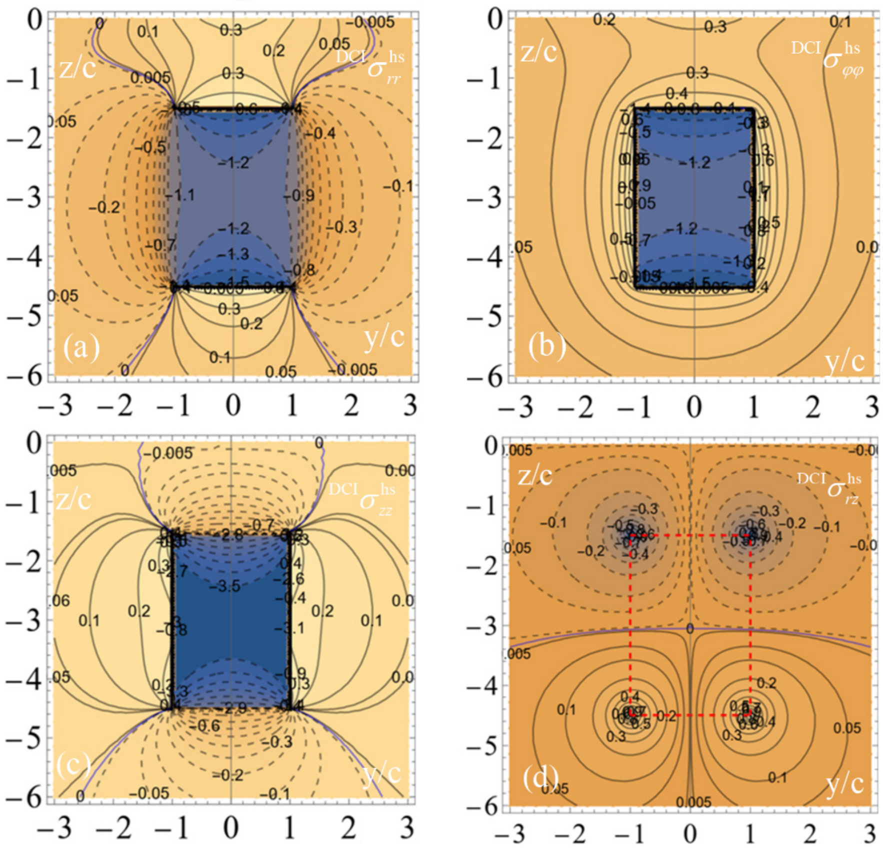

Fig. 3. Stress maps for a cylindrical dilatational inclusion in a half-space. The stresses are given in units of $G(1+\nu)\varepsilon^*/(1-\nu)$, and the coordinates are normalized to the cylinder radius c. Parameters used for plots: the cylinder center coordinate is $z_c = -d = -3c$ the height of the cylinder is $h = 3c$, and Poisson ratio is $\nu = 0.234$.

$$ {}^{\mathrm{DCI,\,out}}\sigma_{rz}^{\mathrm{hs}} = G\frac{(1+\nu)\varepsilon^*}{1-\nu}\Big[J^{(4)}(1,1;0) - J^{(5)}(1,1;0) - J^{(6)}(1,1;0) + J^{(7)}(1,1;0) - \frac{2z}{c}\Big(J^{(6)}(1,1;1) - J^{(7)}(1,1;1)\Big)\Big]. \quad (13j) $$

Figure 3 shows stress maps of DCI in its longitudinal section. It also shows that the boundary stress continuity for the DCI is satisfied: ${}^{\mathrm{DCI}}\sigma_{rr}^{\mathrm{hs}}$ is continuous at the lateral surface, ${}^{\mathrm{DCI}}\sigma_{zz}^{\mathrm{hs}}$ is continuous at the end surfaces, and ${}^{\mathrm{DCI}}\sigma_{rz}^{\mathrm{hs}}$ is continuous at all DCI boundaries. These results are similar to those in the work of Kolesnikova et al. [13]. Components ${}^{\mathrm{DCI}}\sigma_{rz}^{\mathrm{hs}}$ and ${}^{\mathrm{DCI}}\sigma_{zz}^{\mathrm{hs}}$ are equal to zero at the free surface, satisfying the condition of the load-free surface [14].

4.4. Hydrostatic stress in the elastic space

The hydrostatic stress inside and outside the cylinder is determined by the following formula:

$$ \Sigma\sigma = {}^{\mathrm{DCI}}\sigma_{rr}^{\mathrm{hs}} + {}^{\mathrm{DCI}}\sigma_{\varphi\varphi}^{\mathrm{hs}} + {}^{\mathrm{DCI}}\sigma_{zz}^{\mathrm{hs}} = 2G\frac{1+\nu}{1-2\nu}\,{}^{\mathrm{DCI}}\Delta. $$

4.4.1. Inside the cylinder ($z_1 < z < z_2$ and $r \le c$)

$$ \Sigma\,{}^{\mathrm{DCI,\,in}}\sigma^{\mathrm{hs}} = {}^{\mathrm{DCI,\,in}}\sigma_{rr}^{\mathrm{hs}} + {}^{\mathrm{DCI,\,in}}\sigma_{\varphi\varphi}^{\mathrm{hs}} + {}^{\mathrm{DCI,\,in}}\sigma_{zz}^{\mathrm{hs}} = 2G\frac{1+\nu}{1-2\nu}\Bigg[\frac{2(2\nu-1)\varepsilon^*}{1-\nu} + \frac{2(1+\nu)\varepsilon^*}{(1-\nu)}(2\nu-1)\Big(J^{(6)}(1,0;0) - J^{(7)}(1,0;0)\Big)\Bigg]. \quad (14a) $$

4.4.2. Outside the cylinder ($z < z_1$, or $z > z_2$, or $z_1 < z < z_2$ and $r > c$)

$$ \Sigma\,{}^{\mathrm{DCI,\,out}}\sigma^{\mathrm{hs}} = {}^{\mathrm{DCI,\,out}}\sigma_{rr}^{\mathrm{hs}} + {}^{\mathrm{DCI,\,out}}\sigma_{\varphi\varphi}^{\mathrm{hs}} + {}^{\mathrm{DCI,\,out}}\sigma_{zz}^{\mathrm{hs}} = 4G\frac{(1+\nu)^2\varepsilon^*}{1-\nu}\Big[J^{(6)}(1,0;0) - J^{(7)}(1,0;0)\Big]. \quad (14b) $$

4.5. The strain energy of the DCI

The strain energy of the DCI can be found by the general Mura's formula [2]

$$ E = -\frac{1}{2}\int_{({}^{\mathrm{DCI}}V)} {}^{\mathrm{DCI,\,in}}\sigma_{ij}^{\mathrm{hs}}\varepsilon_{ij}^{*}\,dV = -\frac{1}{2}\int_{({}^{\mathrm{DCI}}V)}\Big({}^{\mathrm{DCI,\,in}}\sigma_{rr}^{\mathrm{hs}}\varepsilon_{rr}^{*} + {}^{\mathrm{DCI,\,in}}\sigma_{\varphi\varphi}^{\mathrm{hs}}\varepsilon_{\varphi\varphi}^{*} + {}^{\mathrm{DCI,\,in}}\sigma_{zz}^{\mathrm{hs}}\varepsilon_{zz}^{*}\Big)dV $$

$$ = 2G\frac{1+\nu}{1-\nu}\varepsilon^{*2}\,{}^{\mathrm{DCI}}V + 2G\frac{(1+\nu)^2\varepsilon^{*2}}{(1-\nu)}\int_{({}^{\mathrm{DCI}}V)}\Big(J^{(6)}(1,0;0) - J^{(7)}(1,0;0)\Big)dV. \quad (15) $$

Calculating $\int_{({}^{\mathrm{DCI}}V)}\Big(J^{(6)}(1,0;0) - J^{(7)}(1,0;0)\Big)dV$ in Eq. (15) (see Appendix C), we get the expression for strain energy of DCI:

$$ \begin{aligned} E = {} & 2G\frac{1+\nu}{1-\nu}\varepsilon^{*2}\,{}^{\mathrm{DCI}}V + \frac{4}{3}G\frac{(1+\nu)^2\varepsilon^{*2}}{(1-\nu)}\Bigg[4z_2\left(c^2 - z_2^2\right)E\left(-\frac{c^2}{z_2^2}\right) + 4z_1\left(c^2 - z_1^2\right)E\left(-\frac{c^2}{z_1^2}\right) \\ & + (z_1+z_2)\left(-4c^2 + (z_1+z_2)^2\right)E\left(-\frac{4c^2}{(z_1+z_2)^2}\right) - (z_1+z_2)\left(4c^2 + (z_1+z_2)^2\right)K\left(-\frac{4c^2}{(z_1+z_2)^2}\right) \\ & + 4z_2\left(c^2 + z_2^2\right)K\left(-\frac{c^2}{z_2^2}\right) + 4z_1\left(c^2 + z_1^2\right)K\left(-\frac{c^2}{z_1^2}\right)\Bigg], \end{aligned} \quad (16) $$

where ${}^{\mathrm{DCI}}V = \pi c^2 h$ is volume of the cylinder, $K(m)$ is the complete elliptic integral of the first kind, and $E(m)$ is the complete elliptic integral of the second kind. The factor $2G\frac{1+\nu}{1-\nu}\varepsilon^{*2}\,{}^{\mathrm{DCI}}V$ is the elastic strain energy for the case of an infinite space [13]. The second term is the correction factor due to the free surface [2].

5. DISCUSSION

In this study, we investigated the elasticity boundary-value problem for a dilatational cylindrical inclusion placed in a vicinity of the free surface of an elastically isotropic half-space. To solve this problem, we used an axially symmetric DCI model with the eigenstrain defined by Eq. (3) and the elastic field of each ITDD is determined by the Eqs. (5). The correctness of the found elastic fields of cylinder in the elastic half-space given by Eqs. (8–13) is checked by comparing to the case of infinite space, where the hydrostatic stress on the outside of the cylinder is zero [13]. There is hydrostatic stress in the elastic half-space Eqs (14), similar to the elastic field of ITDD in the half-space [10]. From Eqs. (8–13) it follows that the elastic field in the medium depends on the position of the cylinder relative to the free surface and the size of the cylinder. The

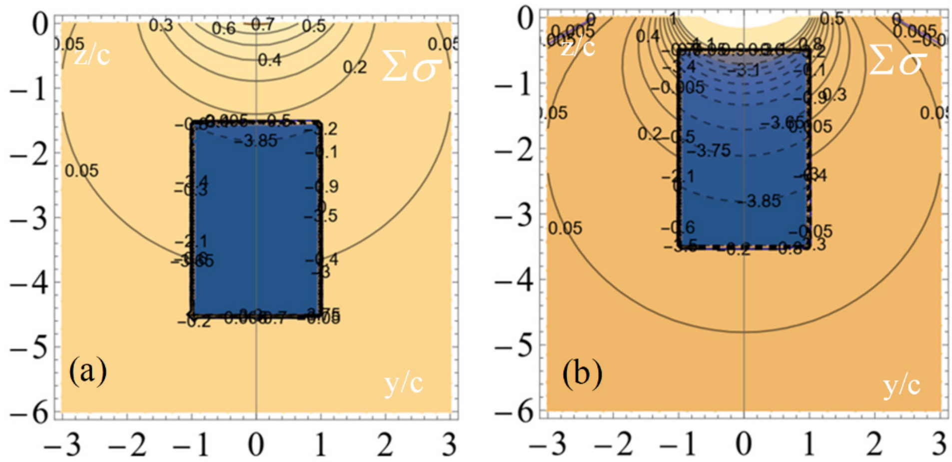

Fig. 4. Maps of hydrostatic stress in the *zy*-plane for the cylindrical inclusion located in elastically isotropic half-space. Stress is given in units of $G(1+\nu)\varepsilon^* / (1-\nu)$ and the coordinates are normalized to the disk radius *c*. Parameters of calculations: the height of the cylinder is $h = 3c$, cylinder coordinate (a) $z_c = -d = -3c$ and (b) $z_c = -d = -2c$; Poisson ratio $\nu = 0.234$.

results also show that, when approaching the free surface, the influence of the free surface on the elastic field of DCI increases. The hydrostatic stress at points near the free surface increases as inclusion approaches the free surface; see Fig. 4. From Eq. (6) it is clear that the strain energy of DCI depends on the position and size of DCI.

6. SUMMARY AND CONCLUSIONS

We have found analytical solutions to the problem of determining the elastic field of a DCI in an elastically isotropic half-space and its strain energy. The results are presented in form of Lipschitz-Hankel integral which is concise and easy to calculate through the expressions containing the Elliptic function. The obtained results demonstrate that when an inclusion is placed in an elastic half-space, hydrostatic stress occurs and depends on the relative position between the cylinder and the free surface. This affects the mechanical and physical properties of the material, which has been proven by many studies [8].

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APPENDIX A. Integrals of Lipschitz-Hankel integrals $J^{(l)}$ $(m, n; p)$ from z_1 to z_2 in the general case in terms of *m*, *n*, and *p*

$$\int_{z_1}^{z_2} J^{(1)}(m,n;p)\,dz_0 = \int_0^\infty J_m(\kappa) J_n\left(\kappa\frac{r}{c}\right)\kappa^p d\kappa \int_{z_1}^{z_2} e^{-\frac{|z-z_0|}{c}\kappa} dz_0$$

$$= c\int_0^\infty J_m(\kappa) J_n\left(\kappa\frac{r}{c}\right)\kappa^{p-1} d\kappa \begin{cases} -e^{-\frac{|z-z_1|}{c}\kappa} + e^{-\frac{|z-z_2|}{c}\kappa}, & (z > z_2), \\ 2 - e^{-\frac{|z-z_1|}{c}\kappa} - e^{-\frac{|z-z_2|}{c}\kappa}, & (z_1 < z < z_2), \\ e^{-\frac{|z-z_1|}{c}\kappa} - e^{-\frac{|z-z_2|}{c}\kappa}, & (z < z_1) \end{cases}$$

$$= c\begin{cases} -J^{(4)}(m,n;p-1) + J^{(5)}(m,n;p-1), & (z > z_2), \\ 2\int_0^\infty J_m(\kappa) J_n\left(\kappa\frac{r}{c}\right)\kappa^{p-1} d\kappa - J^{(4)}(m,n;p-1) - J^{(5)}(m,n;p-1), & (z_1 < z < z_2), \\ J^{(4)}(m,n;p-1) - J^{(5)}(m,n;p-1), & (z < z_1), \end{cases} \tag{A1}$$

$$\int_{z_1}^{z_2} J^{(2)}(m,n;p)\,dz_0 = \int_0^\infty J_m(\kappa) J_n\left(\kappa\frac{r}{c}\right)\kappa^p d\kappa \int_{z_1}^{z_2} e^{-\frac{|z+z_0|}{c}\kappa} dz_0$$

$$= c\int_0^\infty J_m(\kappa) J_n\left(\kappa\frac{r}{c}\right)\kappa^{p-1} d\kappa \begin{cases} -e^{-\frac{|z+z_1|}{c}\kappa} + e^{-\frac{|z+z_2|}{c}\kappa}, & 0 > z > z_2, \\ -e^{-\frac{|z+z_1|}{c}\kappa} + e^{-\frac{|z+z_2|}{c}\kappa}, & z_1 < z < z_2 < 0, \\ -e^{-\frac{|z+z_1|}{c}\kappa} + e^{-\frac{|z+z_2|}{c}\kappa}, & z < z_1 < 0, \end{cases}$$

$$= c\begin{cases} -J^{(6)}(m,n;p-1) + J^{(7)}(m,n;p-1), & 0 > z > z_2, \\ -J^{(6)}(m,n;p-1) + J^{(7)}(m,n;p-1), & z_1 < z < z_2 < 0, \\ -J^{(6)}(m,n;p-1) + J^{(7)}(m,n;p-1), & z < z_1 < 0, \end{cases} \tag{A2}$$

$$\int_{z_1}^{z_2} J^{(3)}(m,n;p)\,dz_0 = \int_0^\infty J_m(\kappa) J_n\left(\kappa\frac{r}{c}\right)\kappa^p d\kappa \int_{z_1}^{z_2} e^{-\frac{|z|+|z_0|}{c}\kappa} dz_0$$

$$= c\int_0^\infty J_m(\kappa) J_n\left(\kappa\frac{r}{c}\right)\kappa^{p-1} d\kappa \begin{cases} -e^{-\frac{|z+z_1|}{c}\kappa} + e^{-\frac{|z+z_2|}{c}\kappa}, & 0 > z > z_2, \\ -e^{-\frac{|z+z_1|}{c}\kappa} + e^{-\frac{|z+z_2|}{c}\kappa}, & z_1 < z < z_2 < 0, \\ -e^{-\frac{|z+z_1|}{c}\kappa} + e^{-\frac{|z+z_2|}{c}\kappa}, & z < z_1 < 0 \end{cases}$$

$$= c\begin{cases} -J^{(6)}(m,n;p-1) + J^{(7)}(m,n;p-1), & 0 > z > z_2, \\ -J^{(6)}(m,n;p-1) + J^{(7)}(m,n;p-1), & z_1 < z < z_2 < 0, \\ -J^{(6)}(m,n;p-1) + J^{(7)}(m,n;p-1), & z < z_1 < 0, \end{cases} \tag{A3}$$

$$\int_{z_1}^{z_2} \operatorname{sgn}(z-z_0) J^{(1)}(m,n;p)\,dz_0 = c\int_0^\infty J_m(\kappa) J_n\left(\kappa\frac{r}{c}\right)\kappa^{p-1} d\kappa \begin{cases} -e^{-\frac{|z-z_1|\kappa}{c}} + e^{-\frac{|z-z_2|\kappa}{c}}, & z > z_2, \\ -e^{-\frac{|z-z_1|\kappa}{c}} + e^{-\frac{|z-z_2|\kappa}{c}}, & z_1 < z < z_2 < 0, \\ -e^{-\frac{|z-z_1|\kappa}{c}} + e^{-\frac{|z-z_2|\kappa}{c}}, & z < z_1 \end{cases}$$

$$= c\begin{cases} -J^{(4)}(m,n;p-1) + J^{(5)}(m,n;p-1), & z > z_2, \\ -J^{(4)}(m,n;p-1) + J^{(5)}(m,n;p-1), & z_1 < z < z_2 < 0, \\ -J^{(4)}(m,n;p-1) + J^{(5)}(m,n;p-1), & z < z_1, \end{cases} \tag{A4}$$

$$\int_{z_1}^{z_2} \operatorname{sgn}(z+z_0) J^{(2)}(m,n;p)\,dz_0 = c\int_0^\infty J_m(\kappa) J_n\left(\kappa\frac{r}{c}\right)\kappa^{p-1} d\kappa \begin{cases} e^{-\frac{|z+z_1|}{c}\kappa} - e^{-\frac{|z+z_2|}{c}\kappa}, & z > z_2, \\ e^{-\frac{|z+z_1|}{c}\kappa} - e^{-\frac{|z+z_2|}{c}\kappa}, & z_1 < z < z_2 < 0, \\ e^{-\frac{|z+z_1|}{c}\kappa} - e^{-\frac{|z+z_2|}{c}\kappa}, & z < z_1, \end{cases}$$

$$= c\begin{cases} J^{(6)}(m,n;p-1) - J^{(7)}(m,n;p-1), & z > z_2, \\ J^{(6)}(m,n;p-1) - J^{(7)}(m,n;p-1), & z_1 < z < z_2 < 0, \\ J^{(6)}(m,n;p-1) - J^{(7)}(m,n;p-1), & z < z_1. \end{cases} \tag{A5}$$

Here $\xi_4 = \frac{|z-z_1|}{c}$, $\xi_5 = \frac{|z-z_2|}{c}$, $\xi_6 = \frac{|z+z_1|}{c}$, $\xi_7 = \frac{|z+z_2|}{c}$, and $z_1 = -\left(d+\frac{h}{2}\right)$, $z_2 = -\left(d-\frac{h}{2}\right)$.

APPENDIX B. Detailed expression of strains inside and outside DCI

B.1. Inside the cylinder ($z_1 < z < z_2$ and $r \le c$)

$$
\begin{aligned}
{}^{\text{DCI, in}}\varepsilon_{rr}^{\text{hs}} = \frac{(1+\nu)\varepsilon^*}{2(1-\nu)}\Big[& 1+\frac{c}{r}J^{(4)}(1,1;-1)-J^{(4)}(1,0;0)+\frac{1}{r}J^{(5)}(1,1;-1)-J^{(5)}(1,0;0) \\
& -\frac{c}{r}(4\nu-3)J^{(6)}(1,1;-1)+(4\nu-3)J^{(6)}(1,0;0) \\
& +\frac{c}{r}(4\nu-3)J^{(7)}(1,1;-1)-(4\nu-3)J^{(7)}(1,0;0)+\frac{2z}{r}J^{(6)}(1,1;0) \\
& -\frac{2z}{c}J^{(6)}(1,0;1)-\frac{2z}{r}J^{(7)}(1,1;0)+\frac{2z}{c}J^{(7)}(1,0;1)-\frac{2(1-\nu)}{(1+\nu)}\Big],
\end{aligned} \tag{B1}
$$

$$
\begin{aligned}
{}^{\text{DCI, in}}\varepsilon_{\varphi\varphi}^{\text{hs}} = \frac{(1+\nu)\varepsilon^*}{2(1-\nu)}\Big[& 1-\frac{c}{r}J^{(4)}(1,1;-1)-\frac{c}{r}J^{(5)}(1,1;-1)+\frac{c}{r}(4\nu-3)J^{(6)}(1,1;-1) \\
& -\frac{c}{r}(4\nu-3)J^{(7)}(1,1;-1)-\frac{2z}{r}J^{(6)}(1,1;0)+\frac{2z}{r}J^{(7)}(1,1;0)-\frac{2(1-\nu)}{(1+\nu)}\Big],
\end{aligned} \tag{B2}
$$

$$
\begin{aligned}
{}^{\text{DCI, in}}\varepsilon_{zz}^{\text{hs}} = \frac{(1+\nu)\varepsilon^*}{2(1-\nu)}\Big[& J^{(4)}(1,0;0)+J^{(5)}(1,0;0)+\frac{2z}{c}J^{(6)}(1,0;1)+\frac{2}{c}J^{(6)}(1,0;0)-2J^{(7)}(1,0;0) \\
& -\frac{2z}{c}J^{(7)}(1,0;1)+(4\nu-3)J^{(6)}(1,0;0)-(4\nu-3)J^{(7)}(1,0;0)-\frac{2(1-\nu)}{(1+\nu)}\Big],
\end{aligned} \tag{B3}
$$

$$
{}^{\text{DCI, in}}\varepsilon_{rz}^{\text{hs}} = \frac{(1+\nu)\varepsilon^*}{2(1-\nu)}\Big[J^{(4)}(1,1;0)-J^{(5)}(1,1;0)-J^{(6)}(1,1;0)+J^{(7)}(1,1;0)-\frac{2z}{c}\big(J^{(6)}(1,1;1)-J^{(7)}(1,1;1)\big)\Big]. \tag{B4}
$$

B.2. Outside the cylinder ($z < z_1$, or $z > z_2$, or $z_1 < z < z_2$ and $r > c$)

For $z < z_1$:

$$
\begin{aligned}
{}^{\text{DCI, out}}\varepsilon_{rr}^{\text{hs}} = \frac{(1+\nu)\varepsilon^*}{2(1-\nu)}\Big[& -\frac{c}{r}\big(J^{(4)}(1,1;-1)--J^{(5)}(1,1;-1)\big)+J^{(4)}(1,0;0)-J^{(5)}(1,0;0) \\
& -(4\nu-3)\frac{c}{r}\big(J^{(6)}(1,1;-1)-J^{(7)}(1,1;-1)\big)+(4\nu-3)\big(J^{(6)}(1,0;0)-J^{(7)}(1,0;0)\big) \\
& +\frac{2z}{r}\big(J^{(6)}(1,1;0)-J^{(7)}(1,1;0)\big)-\frac{2z}{c}\big(J^{(6)}(1,0;1)-J^{(7)}(1,0;1)\big)\Big],
\end{aligned} \tag{B5}
$$

$$
\begin{aligned}
{}^{\text{DCI, out}}\varepsilon_{\varphi\varphi}^{\text{hs}} = \frac{(1+\nu)\varepsilon^*}{2(1-\nu)}\Big[& \frac{c}{r}\big(J^{(4)}(1,1;-1)-J^{(5)}(1,1;-1)\big)+\frac{c}{r}(4\nu-3)\big(J^{(6)}(1,1;-1)-J^{(7)}(1,1;-1)\big) \\
& -\frac{2z}{r}\big(J^{(6)}(1,1;0)-J^{(7)}(1,1;0)\big)\Big],
\end{aligned} \tag{B6}
$$

$$
\begin{aligned}
{}^{\text{DCI, out}}\varepsilon_{zz}^{\text{hs}} = \frac{(1+\nu)\varepsilon^*}{2(1-\nu)}\Big[& -J^{(4)}(1,0;0)+J^{(5)}(1,0;0)+\frac{2z}{c}\big(J^{(6)}(1,0;1)-J^{(7)}(1,0;1)\big) \\
& +2\big(J^{(6)}(1,0;0)-J^{(7)}(1,0;0)\big)+(4\nu-3)\big(J^{(6)}(1,0;0)-J^{(7)}(1,0;0)\big)\Big].
\end{aligned} \tag{B7}
$$

For $z > z_2$:

$$ {}^{\text{DCI, out}}\varepsilon_{rr}^{\text{hs}} = \frac{(1+\nu)\varepsilon^*}{2(1-\nu)}\Big[\frac{c}{r}\Big(J^{(4)}(1,1;-1) - J^{(5)}(1,1;-1)\Big) - J^{(4)}(1,0;0) + J^{(5)}(1,0;0) $$
$$ -(4\nu-3)\frac{c}{r}\Big(J^{(6)}(1,1;-1) - J^{(7)}(1,1;-1)\Big) + (4\nu-3)\Big(J^{(6)}(1,0;0) - J^{(7)}(1,0;0)\Big) $$
$$ + \frac{2z}{r}\Big(J^{(6)}(1,1;0) - J^{(7)}(1,1;0)\Big) - \frac{2z}{c}\Big(J^{(6)}(1,0;1) - J^{(7)}(1,0;1)\Big)\Big], \quad \text{(B8)} $$

$$ {}^{\text{DCI, out}}\varepsilon_{\varphi\varphi}^{\text{hs}} = \frac{(1+\nu)\varepsilon^*}{2(1-\nu)}\Big[-\frac{c}{r}J^{(4)}(1,1;-1) + \frac{c}{r}J^{(5)}(1,1;-1) + \frac{c}{r}(4\nu-3)\Big(J^{(6)}(1,1;-1) - J^{(7)}(1,1;-1)\Big) $$
$$ -\frac{2z}{r}\Big(J^{(6)}(1,1;0) - J^{(7)}(1,1;0)\Big)\Big], \quad \text{(B9)} $$

$$ {}^{\text{DCI, out}}\varepsilon_{zz}^{\text{hs}} = \frac{(1+\nu)\varepsilon^*}{2(1-\nu)}\Big[J^{(4)}(1,0;0) - J^{(5)}(1,0;0) + \frac{2z}{c}\Big(J^{(6)}(1,0;1) - J^{(7)}(1,0;1)\Big) $$
$$ +2\Big(J^{(6)}(1,0;0) - J^{(7)}(1,0;0)\Big) + (4\nu-3)\Big(J^{(6)}(1,0;0) - J^{(7)}(1,0;0)\Big)\Big]. \quad \text{(B10)} $$

For $z_1 < z < z_2$ and $r > c$:

$$ {}^{\text{DCI, out}}\varepsilon_{rr}^{\text{hs}} = \frac{(1+\nu)\varepsilon^*}{2(1-\nu)}\Big[-\frac{c^2}{r^2} + \frac{c}{r}\Big(J^{(4)}(1,1;-1) + J^{(5)}(1,1;-1)\Big) - J^{(4)}(1,0;0) - J^{(5)}(1,0;0) $$
$$ -(4\nu-3)\frac{c}{r}\Big(J^{(6)}(1,1;-1) - J^{(7)}(1,1;-1)\Big) + (4\nu-3)\Big(J^{(6)}(1,0;0) - J^{(7)}(1,0;0)\Big) $$
$$ + \frac{2z}{r}\Big(J^{(6)}(1,1;0) - J^{(7)}(1,1;0)\Big) - \frac{2z}{c}\Big(J^{(6)}(1,0;1) - J^{(7)}(1,0;1)\Big)\Big], \quad \text{(B11)} $$

$$ {}^{\text{DCI, out}}\varepsilon_{\varphi\varphi}^{\text{hs}} = \frac{(1+\nu)\varepsilon^*}{2(1-\nu)}\Big[\frac{c^2}{r^2} - \frac{c}{r}\Big(J^{(4)}(1,1;-1) + J^{(5)}(1,1;-1)\Big) + \frac{c}{r}(4\nu-3)\Big(J^{(6)}(1,1;-1) - J^{(7)}(1,1;-1)\Big) $$
$$ -\frac{2z}{r}\Big(J^{(6)}(1,1;0) - J^{(7)}(1,1;0)\Big)\Big], \quad \text{(B12)} $$

$$ {}^{\text{DCI, out}}\varepsilon_{zz}^{\text{hs}} = \frac{(1+\nu)\varepsilon^*}{2(1-\nu)}\Big[J^{(4)}(1,0;0) + J^{(5)}(1,0;0) + \frac{2z}{c}\Big(J^{(6)}(1,0;1) - J^{(7)}(1,0;1)\Big) $$
$$ +2\Big(J^{(6)}(1,0;0) - J^{(7)}(1,0;0)\Big) + (4\nu-3)\Big(J^{(6)}(1,0;0) - J^{(7)}(1,0;0)\Big)\Big]. \quad \text{(B13)} $$

For $\left(z < z_1,\ \text{or}\ z > z_2,\ \text{or}\ z_1 < z < z_2\ \text{and}\ r > c\right)$:

$$ {}^{\text{DCI}}\varepsilon_{rz}^{\text{hs}} = \frac{(1+\nu)\varepsilon^*}{2(1-\nu)}\Big[J^{(4)}(1,1;0) - J^{(5)}(1,1;0) - J^{(6)}(1,1;0) + J^{(7)}(1,1;0) - \frac{2z}{c}\Big(J^{(6)}(1,1;1) - J^{(7)}(1,1;1)\Big)\Big], \quad \text{(B14)} $$

APPENDIX C. Procedures for integrating $J^{(6)}(1,0;0)$ and $J^{(7)}(1,0;0)$ in the volume ${}^{\text{dci}}V$ of the cylinder

$$ \int_{({}^{\text{DCI}}V)} J^{(6)}(1,0;0)\,dV = \int_0^\infty J_1(\kappa)\,d\kappa \int_{({}^{\text{DCI}}V)} J_0\left(\kappa\frac{r}{c}\right) e^{-\frac{|z+z_1|}{c}\kappa}\,dV, $$

where $dV = r\,dr\,d\varphi\,dz$ is volume element in the cylindrical coordinate system with $\varphi : 0 \to 2\pi$, $r : 0 \to c$, and $z : z_1 \to z_2$.

$$\int_{({}^{\mathrm{DCI}}V)} J^{(6)}(1,0;0)\,dV = \int_0^\infty J_1(\kappa)\,d\kappa \int_{z_1}^{z_2}\int_0^{2\pi}\int_0^c J_0\left(\kappa\frac{r}{c}\right) e^{-\frac{|z+z_1|}{c}\kappa}\, r\,dr\,d\theta\,dz = \int_0^\infty J_1(\kappa)\,d\kappa \int_{z_1}^{z_2}\int_0^c J_0\left(\kappa\frac{r}{c}\right) e^{-\frac{|z+z_1|}{c}\kappa}\, 2\pi r\,dr\,dz$$

$$= 2\pi\int_0^\infty J_1(\kappa)\,d\kappa \int_{z_1}^{z_2} e^{-\frac{|z+z_1|}{c}\kappa}\,dz \int_0^c J_0\left(\kappa\frac{r}{c}\right) r\,dr = 2\pi c^3\int_0^\infty J_1^2(\kappa)\kappa^{-2}\,d\kappa\left(-e^{-\frac{2|z_1|}{c}\kappa} + e^{-\frac{|z_1+z_2|}{c}\kappa}\right)$$

$$= c^2\pi(-z_1+z_2) + \frac{8}{3}z_1\left(c^2 - z_1^2\right)E\left(-\frac{c^2}{z_1^2}\right) + \frac{1}{3}(z_1+z_2)\left(-4c^2 + (z_1+z_2)^2\right)E\left(-\frac{4c^2}{(z_1+z_2)^2}\right)$$

$$+\frac{8}{3}z_1\left(c^2 + z_1^2\right)K\left(-\frac{c^2}{z_1^2}\right) - \frac{1}{3}(z_1+z_2)\left(4c^2 + (z_1+z_2)^2\right)K\left(-\frac{4c^2}{(z_1+z_2)^2}\right), \qquad \text{(C1)}$$

$$\int_{({}^{\mathrm{DCI}}V)} J^{(7)}(1,0;0)\,dV = \int_0^\infty J_1(\kappa)\,d\kappa \int_{({}^{\mathrm{DCI}}V)} J_0\left(\kappa\frac{r}{c}\right) e^{-\frac{|z+z_2|}{c}\kappa}\,dV$$

$$= 2\pi\int_0^\infty J_1(\kappa)\,d\kappa \int_{z_1}^{z_2}\int_0^c J_0\left(\kappa\frac{r}{c}\right) e^{-\frac{|z+z_2|}{c}\kappa}\, r\,dr\,dz = 2\pi c^3\int_0^\infty J_1^2(\kappa)\kappa^{-2}\,d\kappa\left(\mathrm{e}^{-\frac{2|z_2|\kappa}{c}} - \mathrm{e}^{-\frac{|z_1+z_2|\kappa}{c}}\right)$$

$$= c^2\pi(-z_1+z_2) + \frac{8}{3}z_2\left(-c^2 + z_2^2\right)E\left(-\frac{c^2}{z_2^2}\right) - \frac{1}{3}(z_1+z_2)\left(-4c^2 + (z_1+z_2)^2\right)E\left(-\frac{4c^2}{(z_1+z_2)^2}\right)$$

$$-\frac{8}{3}z_2\left(c^2 + z_2^2\right)K\left(-\frac{c^2}{z_2^2}\right) + \frac{1}{3}(z_1+z_2)\left(4c^2 + (z_1+z_2)^2\right)K\left(-\frac{4c^2}{(z_1+z_2)^2}\right). \qquad \text{(C2)}$$

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